

A Note on Total and Paired Domination of Cartesian Product Graphs

K. Choudhary

Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
Kanpur, India

keerti.india@gmail.com

S. Margulies

Department of Mathematics
Pennsylvania State University
State College, PA

margulies@math.psu.edu

I. V. Hicks

Department of Computational and Applied Mathematics
Rice University
Houston, TX

ivhicks@rice.edu

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Abstract

A dominating set D for a graph G is a subset of $V(G)$ such that any vertex not in D has at least one neighbor in D . The domination number $\gamma(G)$ is the size of a minimum dominating set in G . Vizing's conjecture from 1968 states that for the Cartesian product of graphs G and H , $\gamma(G)\gamma(H) \leq \gamma(G \square H)$, and Clark and Suen (2000) proved that $\gamma(G)\gamma(H) \leq 2\gamma(G \square H)$. In this paper, we modify the approach of Clark and Suen to prove a variety of similar bounds related to total and paired domination, and also extend these bounds to the n -Cartesian product of graphs A^1 through A^n .

1 Introduction

We consider simple undirected graphs $G = (V, E)$ with vertex set V and edge set E . The open neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and closed neighborhood by $N_G[v]$. A dominating set D of a graph G is a subset of $V(G)$ such that for all v , $N_G[v] \cap D \neq \emptyset$. A γ -set of G is a minimum dominating set for G , and its size is denoted $\gamma(G)$. A total dominating set D of a graph G is a subset of $V(G)$ such that for all v , $N_G(v) \cap D \neq \emptyset$. A γ_t -set of G is a minimum total dominating set for G , and its size is denoted $\gamma_t(G)$. A paired dominating set D for a graph G is a dominating set such that the subgraph of G induced by D (denoted $G[D]$) has a perfect matching. A γ_{pr} -set of G

is a minimum paired dominating set for G , and its size is denoted $\gamma_{pr}(G)$. In general, for a graph containing no isolated vertices, $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$.

The Cartesian product graph, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$, where vertices gh and $g'h'$ are adjacent whenever $g = g'$ and $(h, h') \in E(H)$, or $h = h'$ and $(g, g') \in E(G)$. Just as the Cartesian product of graphs G and H is denoted $G \square H$, the n -product of graphs A^1, A^2, \dots, A^n is denoted as $A^1 \square A^2 \square \dots \square A^n$, and has vertex set $V(A^1) \times V(A^2) \times \dots \times V(A^n)$, where vertices $u^1 \dots u^n$ and $v^1 \dots v^n$ are adjacent if and only if for some i , $(u^i, v^i) \in E(A^i)$, and $u^j = v^j$ for all other indices $j \neq i$.

Vizing's conjecture from 1968 states that $\gamma(G)\gamma(H) \leq \gamma(G \square H)$. For a thorough review of the activity on this famous open problem, see [1] and references therein. In 2000, Clark and Suen [2] proved that $\gamma(G)\gamma(H) \leq 2\gamma(G \square H)$ by a sophisticated double-counting argument which involved projecting a γ -set of the product graph $G \square H$ down onto the graph H . In this paper, we slightly modify the Clark and Suen double-counting approach and instead project subsets of $G \square H$ down onto both graphs G and H , which allow us to prove five theorems relating to total and paired domination. In this section, we state the results, and in Section 2, we prove the results.

Theorem 1. *Given graphs G and H containing no isolated vertices,*

$$\max\{\gamma(G)\gamma_t(H), \gamma_t(G)\gamma(H)\} \leq 2\gamma(G \square H) .$$

In 2008, Ho [3] proved an inequality for total domination analogous to the Clark and Suen inequality for domination. In particular, Ho proved $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \square H)$. We provide a slightly different proof of Ho's inequality, and then extend the result to the n -product case.

Theorem 2 (Ho [3]). *Given graphs G and H containing no isolated vertices,*

$$\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \square H) .$$

Theorem 3. *Given graphs A^1, A^2, \dots, A^n containing no isolated vertices,*

$$\prod_{i=1}^n \gamma_t(A^i) \leq n\gamma_t(A^1 \square A^2 \square \dots \square A^n) .$$

In 2010, Hou and Jiang [4] proved that $\gamma_{pr}(G)\gamma_{pr}(H) \leq 7\gamma_{pr}(G \square H)$, for graphs G and H containing no isolated vertices. We provide an improvement to this result, and extend the result to the n -product graph.

Theorem 4. *Given graphs G and H containing no isolated vertices,*

$$\gamma_{pr}(G)\gamma_{pr}(H) \leq 6\gamma_{pr}(G \square H) .$$

Theorem 5. *Given graphs A^1, \dots, A^n containing no isolated vertices,*

$$\prod_{i=1}^n \gamma_{pr}(A_i) \leq 2^{n-1}(2n-1)\gamma_{pr}(A_1 \square \dots \square A_n) .$$

2 Main Results

We begin by introducing some notation which will be utilized throughout the proofs in this section. Given $S \subseteq V(G \square H)$, the projection of S onto graphs G and H is defined as

$$\begin{aligned}\Phi_G(S) &= \{g \in V(G) \mid \exists h \in V(H) \text{ with } gh \in S\} , \\ \Phi_H(S) &= \{h \in V(H) \mid \exists g \in V(G) \text{ with } gh \in S\} .\end{aligned}$$

In the case of the n -product graph $A^1 \square \cdots \square A^n$, we project a set of vertices in $V(A^1 \square \cdots \square A^n)$ down to a particular graph A_i . Therefore, given $S \subseteq V(A^1 \square \cdots \square A^n)$, we define

$$\Phi_{A^i}(S) = \{a \in V(A^i) \mid \exists u^1 \cdots u^n \in S \text{ with } a = u^i\} .$$

For $gh \in V(G \square H)$, the G -neighborhood and H -neighborhood of gh are defined as follows:

$$\begin{aligned}N_{\mathbf{G} \square H}(gh) &= \{g'h \in V(G \square H) \mid g' \in N_G(g)\} , \\ N_{G \square \mathbf{H}}(gh) &= \{gh' \in V(G \square H) \mid h' \in N_H(h)\} .\end{aligned}$$

Thus, $N_{\mathbf{G} \square H}(gh)$ and $N_{G \square \mathbf{H}}(gh)$ are both subsets of $V(G \square H)$. Additionally, $E(G \square H)$ can be partitioned into two sets, \mathbf{G} -edges and \mathbf{H} -edges, where

$$\begin{aligned}\mathbf{G}\text{-edges} &= \{(gh, g'h) \in E(G \square H) \mid h \in V(H) \text{ and } (g, g') \in E(G)\} , \\ \mathbf{H}\text{-edges} &= \{(gh, gh') \in E(G \square H) \mid g \in V(G) \text{ and } (h, h') \in E(H)\} .\end{aligned}$$

In the case of the n -product graph $A^1 \square \cdots \square A^n$, we identify the i -neighborhood of a particular vertex, and partition the set of edges $E(A^1 \square \cdots \square A^n)$ into n sets. Thus, we define E_i to be

$$E_i = \left\{ (u^1 \cdots u^n, v^1 \cdots v^n) \mid (u^i, v^i) \in E(A^i), \text{ and } u_j = v_j, \text{ for all other indices } j \neq i \right\} ,$$

and for a vertex $u \in V(A^1 \square \cdots \square A^n)$, we define

$$N_{\square A^i}(u) = \left\{ v \in V(A^1 \square \cdots \square A^n) \mid v \text{ and } u \text{ are connected by } E_i\text{-edge} \right\} .$$

Finally, we need two elementary propositions about matrices that will be utilized throughout the proofs.

Proposition 1. *Let M be a binary matrix. Then either*

- (a) *each column contains a 1, or*
- (b) *each row contains a 0 .*

Prop. 1 refers only to $d_1 \times d_2$ binary matrices. Prop. 2 is a generalization of Prop. 1 for $d_1 \times d_2 \times \cdots \times d_n$ n -ary matrices.

Proposition 2. *Let M be a $d_1 \times d_2 \times \cdots \times d_n$, n -ary matrix (n -ary in this case signifies that M contains entries only in the range $\{1, \dots, n\}$). Then there exists a $j \in \{1, \dots, n\}$ (not necessarily unique), such that each of the $d_1 \times \cdots \times d_{j-1} \times 1 \times d_{j+1} \times \cdots \times d_n$ submatrices of M contains an entry with value j . Such a matrix M is called a j -matrix.*

Note that, given any $d_1 \times d_2 \times \cdots \times d_n$ matrix, there are d_j submatrices of the form $d_1 \times \cdots \times d_{j-1} \times 1 \times d_{j+1} \times \cdots \times d_n$. We will denote such a submatrix as $M[:, i_j, :]$ with $1 \leq i_j \leq d_j$.

Proof. Let M be a $d_1 \times d_2 \times \cdots \times d_n$ n -ary matrix which is not a j -matrix for $1 \leq j \leq n-1$. We will show that M is an n -matrix.

Consider $j = 1$. Since M is not a 1-matrix, there exists at least one $1 \times d_2 \times d_3 \times \cdots \times d_n$ submatrix that does *not* contain a 1. Without loss of generality, let $M[i_1, :]$ with $1 \leq i_1 \leq d_1$ be such a matrix. Next, consider $j = 2$. Since M is also not a 2-matrix, let $M[:, i_2, :]$ with $1 \leq i_2 \leq d_2$ be a $d_1 \times 1 \times d_3 \times \cdots \times d_n$ submatrix that does *not* contain a 2. Therefore, $M[i_1, i_2, :]$ is a $1 \times 1 \times d_3 \times \cdots \times d_n$ submatrix that contains neither a 1 nor a 2. We continue this pattern for $1 \leq j \leq n-1$. Since M is *not* a j -matrix for $1 \leq j \leq n-1$, let $M[i_1, \dots, i_{n-1}, :]$ be the $1 \times \cdots \times 1 \times d_n$ submatrix containing no elements in the set $\{1, \dots, n-1\}$. Therefore, for all $1 \leq x \leq d_n$, $M[i_1, \dots, i_{n-1}, x] = n$, and *all* of the $d_1 \times \cdots \times d_{n-1} \times 1$ submatrices of M contains an entry with value n . Thus, M is an n -matrix. \square

Now, we present the proofs of Theorems 1 through 5.

2.1 Proof of Theorem 1

Proof. Let $\{u_1, \dots, u_{\gamma_t(G)}\}$ be a γ_t -set of G . Partition $V(G)$ into sets $D_1, \dots, D_{\gamma_t(G)}$, such that $D_i \subseteq N_G(u_i)$. Let $\{\bar{u}_1, \dots, \bar{u}_{\gamma(H)}\}$ be a γ -set of H . Partition $V(H)$ into sets $\bar{D}_1, \dots, \bar{D}_{\gamma(H)}$, such that $\bar{u}_j \in \bar{D}_j$ and $\bar{D}_j \subseteq N_H[\bar{u}_j]$. We note that $\{D_1, \dots, D_{\gamma_t(G)}\} \times \{\bar{D}_1, \dots, \bar{D}_{\gamma(H)}\}$ is a partition of $V(G \square H)$. Let D be a γ -set of $G \square H$. Then, for each $gh \notin D$, either $N_{\underline{G} \square \underline{H}}(gh) \cap D$ or $N_{G \square \underline{H}}(gh) \cap D$ is non-empty. Based on this observation, we define the binary $|V(G)| \times |V(H)|$ matrix F such that:

$$F(g, h) = \begin{cases} 1 & \text{if } gh \in D \text{ or } N_{G \square \underline{H}}(gh) \cap D \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Since F is a $|V(G)| \times |V(H)|$ matrix, each of the $D_i \times \bar{D}_j$ subsets of $V(G \square H)$ determines a submatrix of F .

For $i = 1, \dots, \gamma_t(G)$, let $Z_i = D \cap (D_i \times V(H))$, and let

$$S_i = \{\bar{D}_x \mid \text{the submatrix of } F \text{ determined by } D_i \times \bar{D}_x \text{ satisfies Prop. 1a,} \\ \text{with } x \in \{1, \dots, \gamma(H)\}\}.$$

For $j = 1, \dots, \gamma(H)$, let $\overline{Z}_j = D \cap (V(G) \times \overline{D}_j)$, and let

$$\overline{S}_j = \{D_x \mid \text{the submatrix of } F \text{ determined by } D_x \times \overline{D}_j \text{ satisfies Prop. 1b,} \\ \text{with } x \in \{1, \dots, \gamma_t(G)\}\}.$$

Let $d_H = \sum_{i=1}^{\gamma_t(G)} |S_i|$, and $d_G = \sum_{j=1}^{\gamma(H)} |\overline{S}_j|$. Since the partition of $V(G \square H)$ composed of elements $D_i \times \overline{D}_j$ contains $\gamma_t(G)\gamma(H)$ components, and since every $D_i \times \overline{D}_j$ submatrix of F satisfies either conditions (a) or (b) of Prop. 1 (possibly both), $\gamma_t(G)\gamma(H) \leq d_H + d_G$. We will now prove two subclaims which will allow us to bound the size of our various sets.

Claim 1. *If the submatrix of F determined by $D_i \times \overline{D}_j$ satisfies Prop. 1a, then \overline{D}_j is dominated by $\Phi_H(Z_i)$.*

Proof. Let $h \in \overline{D}_j$. We must show that either $h \in \Phi_H(Z_i)$, or h is adjacent to a vertex h' in $\Phi_H(Z_i)$. If $(D_i \times \{h\}) \cap D \neq \emptyset$, there exists a $g \in D_i$ such that $gh \in D$. Thus, $h \in \Phi_H(Z_i)$.

If $(D_i \times \{h\}) \cap D = \emptyset$, then recall that the submatrix of F determined by $D_i \times \overline{D}_j$ satisfies Prop. 1a. Therefore, there is a 1 in every column of the submatrix. This implies there exists a $g \in D_i$ such that $F(g, h) = 1$. Since $gh \notin D$, there exists an $h' \in V(H)$ such that $gh' \in N_{G \square H}(gh) \cap D$. Therefore, (gh', gh) is an **H**-edge, implying $(h, h') \in E(H)$ and h is adjacent to h' . Therefore, \overline{D}_j is dominated by $\Phi_H(Z_i)$. \square

Claim 2. *If the submatrix of F determined by $D_i \times \overline{D}_j$ satisfies Prop. 1b, then D_i is dominated by $\Phi_G(\overline{Z}_j)$. Additionally, $\forall g \in D_i \cap \Phi_G(\overline{Z}_j)$, there exists a vertex $g' \in \Phi_G(\overline{Z}_j)$ such that $(g, g') \in E(G)$.*

We note that this claim does not imply that $\Phi_G(\overline{Z}_j)$ is a total dominating set, but the claim is a slightly stronger condition on domination. When applying this condition, we will say that the set D_i is *non-self dominated* by $\Phi_G(\overline{Z}_j)$.

Proof. The argument for proving that $\Phi_G(\overline{Z}_j)$ dominates D_i is almost identical to the proof of Claim 1. The only difference is that the $D_i \times \overline{D}_j$ submatrix of F satisfies Prop. 1b. Thus, every row contains a 0. But since every vertex in $V(G \square H)$ is dominated by D , this implies that every vertex $g \in D_i$ is dominated by some other (not itself) vertex $g' \in \Phi_G(\overline{Z}_j)$. Thus, D_i is dominated by $\Phi_G(\overline{Z}_j)$, with the slightly stronger condition that *every* vertex in D_i (even those vertices in $D_i \cap \Phi_G(\overline{Z}_j)$) is adjacent to *another* vertex in $\Phi_G(\overline{Z}_j)$. \square

Claim 3. *For $i = 1, \dots, \gamma_t(G)$, $|S_i| \leq |Z_i|$. Similarly, for $j = 1, \dots, \gamma(H)$, $|\overline{S}_j| \leq |\overline{Z}_j|$.*

Proof. Let $S_i = \{\overline{D}_{j_1}, \overline{D}_{j_2}, \dots, \overline{D}_{j_k}\}$, and let $A = \Phi_H(Z_i)$. Note that $|A| \leq |Z_i|$. By Claim 1, A dominates $\cup_{x=1}^k \overline{D}_{j_x}$. Therefore, $A \cup \{\overline{u}_j \mid j \notin \{j_1, j_2, \dots, j_k\}\}$ is a dominating set of H , and, since the sets A and $\{\overline{u}_j \mid j \notin \{j_1, j_2, \dots, j_k\}\}$ are disjoint, then

$$|A \cup \{\overline{u}_j \mid j \notin \{j_1, j_2, \dots, j_k\}\}| = |A| + (\gamma(H) - k) \geq \gamma(H).$$

Hence, $k = |S_i| \leq |A| \leq |Z_i|$.

For the proof of second part, let $\overline{S}_j = \{D_{i_1}, D_{i_2}, \dots, D_{i_k}\}$, and let $A = \Phi_G(\overline{Z}_j)$. Again, note that $|A| \leq |\overline{Z}_j|$. Then by Claim 2, A dominates $\cup_{x=1}^k D_{i_x}$, with the stronger condition that $\forall g \in D_{i_x} \cap A$, there exists a vertex $g' \in A$ such that $(g, g') \in E(G)$. Now we consider $A \cap \{u_i \mid i \notin \{i_1, i_2, \dots, i_k\}\}$. If this intersection is non-empty, let $A \cap \{u_i \mid i \notin \{i_1, i_2, \dots, i_k\}\} = \{u_{i_{k+1}}, \dots, u_{i_l}\}$. Then, A dominates $\cup_{x=1}^l D_{i_x}$ with the same stronger condition. Moreover, the sets A and $\{u_i \mid i \notin \{i_1, i_2, \dots, i_k, \dots, i_l\}\}$ are disjoint.

We claim that $A \cup \{u_i \mid i \notin \{i_1, \dots, i_l\}\}$ is a total dominating set of G . To see this, consider any vertex $g \in V(G)$. If $g \in D_x$ with $x \in \{i_1, i_2, \dots, i_k\}$, then by the stronger condition on domination associated with Claim 2, g is adjacent to another vertex in A . If $g \in D_x$ with $x \notin \{i_1, \dots, i_k\}$, then $u_x \in \{u_i \mid i \notin \{i_1, \dots, i_k\}\}$, and g is adjacent to u_x , since u_x dominates D_x . We note that u_x is either in A (if $k+1 \leq x \leq l$) or in $\{u_i \mid i \notin \{i_1, \dots, i_l\}\}$. In either case, $A \cup \{u_i \mid i \notin \{i_1, \dots, i_l\}\}$ is a total dominating set of G , and

$$|A \cup \{u_i \mid i \notin \{i_1, i_2, \dots, i_l\}\}| = |A| + (\gamma_t(G) - l) \geq \gamma_t(G) .$$

Hence, as before, $k = |\overline{S}_j| \leq l \leq |A| \leq |\overline{Z}_j|$. □

To conclude the proof, we observe that

$$\begin{aligned} d_H &= \sum_{i=1}^{\gamma_t(G)} |S_i| \leq \sum_{i=1}^{\gamma_t(G)} |Z_i| \leq |D| , \\ d_G &= \sum_{j=1}^{\gamma(H)} |\overline{S}_j| \leq \sum_{j=1}^{\gamma(H)} |\overline{Z}_j| \leq |D| . \end{aligned}$$

Hence, $\gamma_t(G)\gamma(H) \leq d_H + d_G \leq 2|D| \leq 2\gamma(G \square H)$. Moreover, we can similarly prove that $\gamma(G)\gamma_t(H) \leq 2\gamma(G \square H)$. Therefore, $\max\{\gamma(G)\gamma_t(H), \gamma_t(G)\gamma(H)\} \leq 2\gamma(G \square H)$. □

2.2 Proof of Theorem 2

Proof. Let $\{u_1, \dots, u_{\gamma_t(G)}\}$ be a γ_t -set of G . Partition $V(G)$ into sets $D_1, \dots, D_{\gamma_t(G)}$, such that if $u \in D_i$ then $u \in N_G(u_i)$ for all $i = 1, \dots, \gamma_t(G)$. Similarly, let $\{\overline{u}_1, \dots, \overline{u}_{\gamma_t(H)}\}$ be a γ_t -set of H and $\overline{D}_1, \dots, \overline{D}_{\gamma_t(H)}$ be the corresponding partitions. Then, $\{D_1, \dots, D_{\gamma_t(G)}\} \times \{\overline{D}_1, \dots, \overline{D}_{\gamma_t(H)}\}$ forms a partition of $V(G \square H)$.

Let D be a γ_t -set of $G \square H$. Then, for each $gh \in V(G \square H)$, either the set $N_{\underline{G} \square H}(gh) \cap D$ or the set $N_{G \square \underline{H}}(gh) \cap D$ is non-empty. Based on this observation, we define the binary $|V(G)| \times |V(H)|$ matrix F :

$$F(g, h) = \begin{cases} 1 & \text{if } N_{G \square \underline{H}}(gh) \cap D \neq \emptyset , \\ 0 & \text{otherwise .} \end{cases}$$

For $i = 1, \dots, \gamma_t(G)$, let $Z_i = D \cap (D_i \times V(H))$, and let

$$\begin{aligned} S_i &= \{\overline{D}_x \mid \text{the submatrix of } F \text{ determined by } D_i \times \overline{D}_x \text{ satisfies Prop. 1a,} \\ &\quad \text{with } x \in \{1, \dots, \gamma_t(H)\}\} . \end{aligned}$$

For $j = 1, \dots, \gamma_t(H)$, let $\overline{Z}_j = D \cap (V(G) \times \overline{D}_j)$, and let

$$\overline{S}_j = \{D_x \mid \text{the submatrix of } F \text{ determined by } D_x \times \overline{D}_j \text{ satisfies Prop. 1b,} \\ \text{with } x \in \{1, \dots, \gamma_t(G)\}\}.$$

Let $d_H = \sum_{i=1}^{\gamma_t(G)} |S_i|$, and $d_G = \sum_{j=1}^{\gamma_t(H)} |\overline{S}_j|$. Since the partition of $V(G \square H)$ composed of elements $D_i \times \overline{D}_j$ contains $\gamma_t(G)\gamma_t(H)$ components, and since every submatrix of F determined by $D_i \times \overline{D}_j$ satisfies either Prop. 1a or 1b (or possibly both), then $\gamma_t(G)\gamma_t(H) \leq d_H + d_G$.

Furthermore, by similar arguments given in the proof of Theorem 1 (specifically, Claims 1 and 2), we can conclude, as before, that for $i = 1, \dots, \gamma_t(G)$, $|S_i| \leq |Z_i|$ and, for $j = 1, \dots, \gamma_t(H)$, $|\overline{S}_j| \leq |\overline{Z}_j|$. Finally,

$$d_H = \sum_{i=1}^{\gamma_t(G)} |S_i| \leq \sum_{i=1}^{\gamma_t(G)} |Z_i| = |D| = \gamma_t(G \square H), \\ d_G = \sum_{j=1}^{\gamma_t(H)} |\overline{S}_j| \leq \sum_{j=1}^{\gamma_t(H)} |\overline{Z}_j| = |D| = \gamma_t(G \square H).$$

Summing these two equations, we see $d_H + d_G \leq 2\gamma_t(G \square H)$, which implies $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \square H)$. \square

2.3 Proof of Theorem 3

Proof. For $i = 1, \dots, n$, let $\{u_1^i, \dots, u_{\gamma_t(A^i)}^i\}$ be a γ_t -set of A^i , and $D_1^i, \dots, D_{\gamma_t(A^i)}^i$ be the corresponding partitions (as defined in the proof of Theorem 2).

Let $Q = \{D_1^1, \dots, D_{\gamma_t(A^1)}^1\} \times \dots \times \{D_1^n, \dots, D_{\gamma_t(A^n)}^n\}$. Then Q forms a partition of $V(A^1 \square \dots \square A^n)$ with $|Q| = \prod_{i=1}^n \gamma_t(A^i)$.

Let D be a γ_t -set of $A^1 \square \dots \square A^n$. Then, for each $u \in V(A^1 \square \dots \square A^n)$, there exists an i such that $N_{\square A^i}(u) \cap D$ is non-empty. Based on this observation (as in the 2-dimensional case), we define an n -ary $|V(A^1)| \times \dots \times |V(A^n)|$ matrix F such that:

$$F(u_1, \dots, u_n) = \min\{i \mid N_{\square A^i}(u_1 \dots u_n) \cap D \neq \emptyset\}.$$

For $j = 1, \dots, n$, let $d_j \subseteq Q$ be the set of the elements in Q which are j -matrices. By Prop. 2, each element of Q belongs to at least one d_j -set. Then, $\prod_{i=1}^n \gamma_t(A^i) \leq \sum_{j=1}^n |d_j|$.

Claim 4. For $j = 1, \dots, n$, $|d_j| \leq |D|$.

Proof. We prove here that $|d_n| \leq |D|$, but a similar proof can be performed for any other j . Similar to Q , let $B = \{D_1^1, \dots, D_{\gamma_t(A^1)}^1\} \times \dots \times \{D_1^{n-1}, \dots, D_{\gamma_t(A^{n-1})}^{n-1}\}$. For convenience,

we denote B as $\{B_1, \dots, B_{|B|}\}$, where $|B| = \prod_{i=1}^{(n-1)} \gamma_t(A^i)$.

For $p = 1, \dots, |B|$, let $Z_p = D \cap (B_p \times A^n)$, and

$$S_p = \{D_x^n \mid \text{the submatrix of } F \text{ determined by } B_p \times D_x^n \text{ is an } n\text{-matrix,} \\ \text{with } x \in \{1, \dots, \gamma_t(A^n)\}\}.$$

Note that if $q \in Q$ is a n -matrix, then the projection of q on A^n is *non-self-dominated* by the projection of D on A^n (the same condition used in Claim 2). Moreover, if q is written as $B_p \times D_x^n$ for some $p \in \{1, \dots, |B|\}$ and $x \in \{1, \dots, \gamma_t(A^n)\}$, then D_x^n is non-self-dominated by the projection of Z_p on A^n .

We now claim that for $p = 1, \dots, |B|$, $|S_p| \leq |Z_p|$. We prove this claim in a manner very similar to the proof of Claim 2. Let $S_p = \{D_{i_1}^n, D_{i_2}^n, \dots, D_{i_t}^n\}$ and let $\Phi_{A^n}(Z_p)$ be the projection of Z_p on A^n . As in Claim 2, $\Phi_{A^n}(Z_p)$ dominates $\cup_{x=1}^t D_{i_x}^n$, and if $\Phi_{A^n}(Z_p) \cap \{u_i^n \mid i \notin \{i_1, i_2, \dots, i_t\}\}$ is non-empty, let $\Phi_{A^n}(Z_p) \cap \{u_i^n \mid i \notin \{i_1, i_2, \dots, i_t\}\} = \{u_{i_{t+1}}^n, \dots, u_{i_l}^n\}$. Then, as before, $\Phi_{A^n}(Z_p) \cup \{u_i^n \mid i \notin \{i_1, i_2, \dots, i_t, \dots, i_l\}\}$ is a total dominating set of A^n , and the sets $\Phi_{A^n}(Z_p)$ and $\{u_i^n \mid i \notin \{i_1, i_2, \dots, i_l\}\}$ are disjoint. Therefore, $|\Phi_{A^n}(Z_p) \cup \{u_i^n \mid i \notin \{i_1, i_2, \dots, i_l\}\}| = |\Phi_{A^n}(Z_p)| + (\gamma_t(A^n) - t) \geq \gamma_t(A^n)$. Hence, $t = |S_p| \leq l \leq |\Phi_{A^n}(Z_p)| \leq |Z_p|$.

$$\text{Now, } |d_n| = \sum_{p=1}^{|B|} |S_p| \leq \sum_{p=1}^{|B|} |Z_p| \leq |D|. \quad \square$$

$$\text{To conclude the proof, } \prod_{i=1}^n \gamma_t(A^i) \leq \sum_{j=1}^n |d_j| \leq n|D| = n\gamma_t(A^1 \square \dots \square A^n). \quad \square$$

2.4 Proof of Theorem 4

Proof. Let $\{x_1, y_1, \dots, x_k, y_k\}$ be a γ_{pr} -set of G , where for each i , $(x_i, y_i) \in E(G)$. Thus, $\gamma_{pr}(G) = 2k$. Partition $V(G)$ into sets D_1, \dots, D_k , such that $\{x_i, y_i\} \subseteq D_i \subseteq N_G[x_i, y_i]$ for $1 \leq i \leq k$. Similarly, let $\{\bar{x}_1, \bar{y}_1, \dots, \bar{x}_l, \bar{y}_l\}$ be a γ_{pr} -set of H , where for each j , $(\bar{x}_j, \bar{y}_j) \in E(H)$. Thus, $\gamma_{pr}(H) = 2l$. Partition $V(H)$ into sets $\bar{D}_1, \dots, \bar{D}_l$, such that $\{\bar{x}_j, \bar{y}_j\} \subseteq \bar{D}_j \subseteq N_H[\bar{x}_j, \bar{y}_j]$ for $1 \leq j \leq l$. Now, $\{D_1, \dots, D_k\} \times \{\bar{D}_1, \dots, \bar{D}_l\}$ forms a partition of $V(G \square H)$.

Let D be a γ_{pr} -set of $G \square H$. Then, for each $gh \notin D$, either $N_{\underline{G} \square H}(gh) \cap D$ or $N_{G \square \underline{H}}(gh) \cap D$ is non-empty. Based on this observation, we define the binary $|V(G)| \times |V(H)|$ matrix F such that:

$$F(g, h) = \begin{cases} 1 & \text{if } gh \in D \text{ or } N_{G \square \underline{H}}(gh) \cap D \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Since D is a γ_{pr} -set, the subgraph of $G \square H$ induced by D has a perfect matching. Thus, D can be written as the disjoint union of

$$D_G = \{gh \in D \mid \text{the matching edge incident to } gh \text{ is a } \mathbf{G}\text{-edge}\}, \text{ and} \\ D_H = \{gh \in D \mid \text{the matching edge incident to } gh \text{ is an } \mathbf{H}\text{-edge}\}.$$

For $i = 1, \dots, k$, let $Z_{G_i} = D_G \cap (D_i \times V(H))$, and $Z_{H_i} = D_H \cap (D_i \times V(H))$. For $j = 1, \dots, l$, let $\overline{Z}_{G_j} = D_G \cap (V(G) \times \overline{D}_j)$, and let $\overline{Z}_{H_j} = D_H \cap (V(G) \times \overline{D}_j)$. By Claims 1 and 2, if the submatrix of F determined by $D_i \times \overline{D}_j$ satisfies Prop. 1a, then \overline{D}_j is dominated by $\Phi_H(Z_{G_i} \cup Z_{H_i})$, and if the submatrix of F determined by $D_i \times \overline{D}_j$ satisfies Prop. 1b, then D_i is dominated by $\Phi_G(\overline{Z}_{G_j} \cup \overline{Z}_{H_j})$.

For $i = 1, \dots, k$, and $j = 1, \dots, l$, let

$$S_i = \{\overline{D}_x \mid \text{the submatrix of } F \text{ determined by } D_i \times \overline{D}_x \text{ satisfies Prop. 1a,} \\ \text{with } x \in \{1, \dots, l\}\}, \\ \overline{S}_j = \{D_x \mid \text{the submatrix of } F \text{ determined by } D_x \times \overline{D}_j \text{ satisfies Prop. 1b,} \\ \text{with } x \in \{1, \dots, k\}\}.$$

Finally, let $d_H = \sum_{i=1}^k |S_i|$, and $d_G = \sum_{j=1}^l |\overline{S}_j|$. Then, as before, $kl \leq d_H + d_G$, since each of the kl submatrices of F determined by $D_i \times \overline{D}_j$ satisfies one (or both) of the conditions of Prop. 1. We now prove a claim that will allow us to bound the sizes of our various sets and conclude the proof.

Claim 5. For $i = 1 \dots, k$, $2|S_i| \leq 2|Z_{G_i}| + |Z_{H_i}|$.

Proof. Let $S_i = \{\overline{D}_{j_1}, \overline{D}_{j_2}, \dots, \overline{D}_{j_t}\}$. Let $A = \Phi_H(Z_{G_i})$, $B = \Phi_H(Z_{H_i})$, and $C = \{\overline{x}_j \mid j \notin \{j_1, j_2, \dots, j_t\}\} \cup \{\overline{y}_j \mid j \notin \{j_1, j_2, \dots, j_t\}\}$.

Let M be the matching on $B \cup C$ formed by taking all of the $\{\overline{x}_j, \overline{y}_j\}$ edges induced by the vertices in C , and then adding the edges from a maximal matching on the remaining unmatched vertices in B . Then, $E = A \cup B \cup C$ is a dominating set of H with M as a matching. Let $M_1 = V(M)$ and $M_2 = (B \cup C) \setminus M_1$. We note that M_1 consists of all the vertices in C plus the matched vertices from B , and M_2 contains only the unmatched vertices from B . Therefore, $|M_1| + 2|M_2| \leq |C| + |Z_{H_i}|$. To see this more clearly, consider a vertex $gh \in Z_{H_i}$ that is matched by an H-edge to a vertex gh' such that $h \notin V(M)$. This implies that either h' coincides with a vertex of C , or h' coincides with the projection of some other vertex of Z_{H_i} (because otherwise h would be matched with h'). Therefore, $2|M_2|$ is equivalent to counting h' , and we see that $|M_1| + 2|M_2| \leq |C| + |Z_{H_i}|$.

In order to obtain a perfect matching of E , we recursively modify E by choosing an unmatched vertex h in E (a vertex in either A or B , since all vertices in C are automatically matched), and then either matching it with an appropriate vertex, or removing it from E . Specifically, if $N_H(h) \setminus V(M)$ is non-empty, there exists a vertex $h' \in N_H(h) \setminus V(M)$ such that we can add h' to E and (h, h') to the matching M . Otherwise, h is incident on only matched vertices, and we can remove h from E without altering the fact that E is a dominating set.

Our recursively modified E (denoted by E_{rec}) is now a paired dominating set of H . Furthermore, in the worst case, we have doubled the unmatched vertices from B , and also doubled the vertices in A . Thus,

$$2l \leq |E_{\text{rec}}| \leq 2|A| + |M_1| + 2|M_2| .$$

Since $|M_1| + 2|M_2| \leq |C| + |Z_{H_i}|$, this implies that $2l - |C| \leq 2|A| + |Z_{H_i}|$. Furthermore, since $2l - |C| = 2|S_i|$, we see that $2|S_i| \leq 2|Z_{G_i}| + |Z_{H_i}|$. \square

Similarly, for $j = 1, \dots, l$, we can show that $2|\bar{S}_j| \leq |\bar{Z}_{G_j}| + 2|\bar{Z}_{H_j}|$. We now see

$$\begin{aligned} 2 \sum_{i=1}^k |S_i| + 2 \sum_{j=1}^l |\bar{S}_j| &\leq 2 \sum_{i=1}^k |Z_{G_i}| + \sum_{i=1}^k |Z_{H_i}| + \sum_{j=1}^l |\bar{Z}_{G_j}| + 2 \sum_{j=1}^l |\bar{Z}_{H_j}| , \\ &\leq \underbrace{\sum_{i=1}^k |Z_{G_i}| + \sum_{i=1}^k |Z_{H_i}|}_D + \underbrace{\sum_{j=1}^l |\bar{Z}_{G_j}| + \sum_{j=1}^l |\bar{Z}_{H_j}|}_D + \underbrace{\sum_{i=1}^k |Z_{G_i}| + \sum_{j=1}^l |\bar{Z}_{H_j}|}_D , \\ &\leq 3|D| . \end{aligned}$$

To conclude the proof, we note that

$$\begin{aligned} 2(d_H + d_G) &= 2 \sum_{i=1}^k |S_i| + 2 \sum_{j=1}^l |\bar{S}_j| \leq 3|D| , \\ 2(kl) &= \gamma_{pr}(G) \frac{\gamma_{pr}(H)}{2} \leq 3|D| , \\ \gamma_{pr}(G) \gamma_{pr}(H) &\leq 6\gamma_{pr}(G \square H) . \end{aligned}$$

\square

2.5 Proof of Theorem 5

Proof. For $i = 1, \dots, n$, let $k_i = \gamma_{pr}(A^i)/2$, and let $\{x_1^i, y_1^i, \dots, x_{k_i}^i, y_{k_i}^i\}$ be a γ_{pr} -set of A^i , and $D_1^i, \dots, D_{k_i}^i$ be the corresponding partitions (as defined in Theorem 4).

Let $Q = \{D_1^1, \dots, D_{k_1}^1\} \times \dots \times \{D_1^n, \dots, D_{k_n}^n\}$. Then Q forms a partition of $V(A^1 \square \dots \square A^n)$ with $|Q| = \prod_{i=1}^n \gamma_{pr}(A^i)/2 = \frac{1}{2^n} \prod_{i=1}^n \gamma_{pr}(A^i)$.

Let D be a γ_{pr} -set of $A^1 \square \dots \square A^n$. Then, for each $u \in V(A^1 \square \dots \square A^n)$, there exists an i such that $N_{\square A^i}(u) \cap D$ is non-empty. We now proceed slightly differently than previously. Based on this observation (as in the 2-dimensional case), we define n different matrices F^i with $i = 1, \dots, n$, where each of the n matrices is an n -ary $|V(A^1)| \times \dots \times |V(A^n)|$ matrix F^i such that:

$$F^i(u_1, \dots, u_n) = \begin{cases} i & \text{if } u_1 \dots u_n \in D , \\ j_{\min} & \text{where } j_{\min} = \min\{ j \mid N_{\square A^j}(u_1 \dots u_n) \cap D \neq \emptyset \} . \end{cases}$$

Thus, each of the n matrices F^i with $i = 1, \dots, n$ differs only in the entries that correspond to vertices in the paired dominating set D .

For $j = 1, \dots, n$ and $i = 1, \dots, n$, let $d_j^i \subseteq Q$ be the set of the elements in Q which are j -matrices in the matrix F^i . By Prop. 2, each element of Q belongs to at least one d_j^i -set for each $i = 1, \dots, n$. Now, if an element $q \in Q$ belongs to the d_j^i -set, then q also belongs to the d_j^i -set. To see this, if M_i and M_j are the submatrices determined by q with respect to the matrices F^i and F^j , respectively, then all the entries that do not match in M_i and M_j have value j in M_j . Thus, each $q \in Q$ belongs to at least one d_i^i -set for some $i \in \{1, \dots, n\}$. Then, $\frac{1}{2^n} \prod_{i=1}^n \gamma_{pr}(A^i) \leq \sum_{i=1}^n |d_i^i|$.

Similar to Q , let $B = \{D_1^1, \dots, D_{k_1}^1\} \times \dots \times \{D_1^{n-1}, \dots, D_{k_{n-1}}^{n-1}\}$. For convenience, we denote B as $\{B_1, \dots, B_{|B|}\}$, where $|B| = \prod_{i=1}^{n-1} \gamma_{pr}(A^i)/2 = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \gamma_{pr}(A^i)$.

Since D is a γ_{pr} -set, the subgraph of $A^1 \square \dots \square A^n$ induced by D has a perfect matching. Let

$$D_i = \{u \in D \mid \text{the matching edge incident to } u \text{ is in } E_i\}.$$

Then, D can be written as the disjoint union of the subsets D_i . For $p = 1, \dots, |B|$ and $i = 1, \dots, n$, let $Z_p^i = D_i \cap (B_p \times A^n)$, and

$$S_p = \{D_x^n \mid \text{the submatrix of } F^n \text{ determined by } B_p \times D_x^n \text{ is an } n\text{-matrix,} \\ \text{with } x \in \{1, \dots, k_n\}\}.$$

Claim 6. For $p = 1, \dots, |B|$, $2|S_p| \leq 2|Z_p^1| + \dots + 2|Z_p^{n-1}| + |Z_p^n|$.

Proof. Let $S_p = \{D_{j_1}^n, D_{j_2}^n, \dots, D_{j_t}^n\}$, and for $j = 1, \dots, n$, let $V_j = \Phi_{A^n}(Z_p^j)$. Note that $|V_j| \leq |Z_p^j|$. Similiar to the proof of Claim 5, let $C = \{x_j^n \mid j \notin \{j_1, j_2, \dots, j_t\}\} \cup \{y_j^n \mid j \notin \{j_1, j_2, \dots, j_t\}\}$.

Let M be the matching on $V_n \cup C$ formed by taking all of the $\{x_j^n, y_j^n\}$ edges induced by the vertices in C , and then adding the edges from a maximal matching on the remaining unmatched vertices in V_n . Then, $E = V_1 \cup \dots \cup V_n \cup C$ is a dominating set of A^n with M as a matching.

Let $M_1 = V(M)$ and $M_2 = (V_n \cup C) \setminus M_1$. We note that M_1 consists of all the vertices in C plus the matched vertices from V_n , and M_2 contains only the unmatched vertices from V_n .

In order to obtain a perfect matching, we recursively modify E by choosing an unmatched vertex a in E , and then either matching it with an appropriate vertex, or removing it from E . Specifically, if $N_{A^n}(a) \setminus V(M)$ is non-empty, there exists a vertex $a' \in N_{A^n}(a) \setminus V(M)$ such that we can add a' to E and (a, a') to the matching M . Otherwise, a is incident on only matched vertices, and we can safely remove it from E without altering the fact that E is a dominating set.

Our recursively modified E (denoted by E_{rec}) is now a paired dominating set of A_n . Furthermore, in the worst case, we have doubled the unmatched vertices from V_n , and also doubled the vertices in V_1, \dots, V_{n-1} . Thus,

$$2k_n \leq |E_{\text{rec}}| \leq 2|V_1| + \dots + 2|V_{n-1}| + |M_1| + 2|M_2| .$$

This implies that $2k_n - |C| \leq 2|V_1| + \dots + 2|V_{n-1}| + |Z_p^n|$. Since $2k_n - |C| = 2|S_p|$, therefore, $2|S_p| \leq 2|V_1| + \dots + 2|V_{n-1}| + |Z_p^n| \leq 2|Z_p^1| + \dots + 2|Z_p^{n-1}| + |Z_p^n|$. \square

To conclude the proof, we follow a similar method as in the proof of Theorem 4. We begin by noting that,

$$|d_n^n| = \sum_{p=1}^{|B|} |S_p| .$$

Using Claim 6, we now see

$$2 \sum_{p=1}^{|B|} |S_p| \leq \sum_{p=1}^{|B|} \left(2 \sum_{j=1}^n |Z_p^j| - |Z_p^n| \right) = 2|D| - \sum_{p=1}^{|B|} |Z_p^n| = 2|D| - |D_n| .$$

Therefore, $2|d_n^n| \leq 2|D| - |D_n|$. Similarly, we can show that $2|d_i^i| \leq 2|D| - |D_i|$ for $i = 1, \dots, n$. To conclude the proof, we see

$$\begin{aligned} \frac{1}{2^{n-1}} \prod_{i=1}^n \gamma_{pr}(A_i) &= 2(k_1 \cdots k_n) \leq 2 \sum_{i=1}^n |d_i^i| \leq 2n|D| - \sum_{i=1}^n |D_i| = (2n-1)|D| , \\ \prod_{i=1}^n \gamma_{pr}(A_i) &\leq 2^{n-1}(2n-1)\gamma_{pr}(A_1 \square \cdots \square A_n) . \end{aligned}$$

\square

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